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Global Linear and Quadratic One-step Smoothing Newton Method for P_0-LCP

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Abstract. We propose a new smoothing Newton method for solving the P_0-matrix linear complementarity problem (P_0-LCP) based on CHKS smoothing function. Our algorithm solves only one linear system of equations and performs only one line search per iteration. It is shown to converge to a P_0-LCP solution globally linearly and locally quadratically without the strict complementarity assumption at the solution. To the best of author's knowledge, this is the first one-step smoothing Newton method to possess both global linear and local quadratic convergence. Preliminary numerical results indicate that the proposed algorithm is promising.

Key words: P_0-matrix linear complementarity problem, smoothing Newton method, global linear convergence, quadratic convergence

1. Introduction

We consider the P_0-matrix linear complementarity problem (P_0-LCP) of finding a vector $(x, y) \in R^n \times R^n$ such that

$$x \geqslant 0, \quad y \geqslant 0, \quad x^T y = 0, \quad Mx + q - y = 0, \tag{1.1}$$

where the matrix $M \in R^{n\times n}$ is a P_0-matrix and the vector $q \in R^n$. A matrix $M \in R^{n\times n}$ is said to be a P_0-matrix if all its principal minors are nonnegative. An LCP is called a P_0-LCP if the involved matrix M is a P_0-matrix. The class of the P_0-LCP includes the monotone LCP and the P-matrix LCP. LCP has various important applications in many fields [12,19]. In the past few decades, it has been studied extensively; e.g., see [2–5,8,9,18,23] and references therein. In this paper, we are interested in developing a smoothing Newton method to solve the P_0-LCP. In general, a smoothing Newton method uses a smooth function to reformulate the problem concerned as a family of parameterized smooth equations, solves the smooth equations approximately by using Newton method per iteration. By reducing the parameter to zero, it is hopeful that a solution of the original problem can be found.

Recently, the idea of using smooth functions to solve the nonsmooth equation reformulation of complementarity problems and related problems has been studied actively. The methods proposed so far can be classified into two related classes: smoothing Newton methods and non-interior continuation or path-following methods. They have attracted a lot of attention due to their convenience and numerical implementation. Indeed, it has been demonstrated that many of them are superior to interior-point methods in terms of numerical performence; see Billups et al. [1]. We review some of the important progresses made related to the methods. Smale [32] first studied the smoothing Newton method for solving linear programming and LCP. Chen and Harker [4] first introduced a non-interior continuation method for solving the LCP with a P_0 and R_0 matrix. They concentrated on establishing the properties of the smoothing path. Later, Kanzow [21] improved the method by refining the smooth function, referred as Chen-Harker-Kanzow-Smale (CHKS) smooth function, which was generalized by Chen and Mangasarian [9], and Gabrial and Moré [15]. Burke and Xu [3] introduced the concept of neighborhood of smoothing path into their continuation method, which allowed their algorithm to establish a global linear convergence result for the monotone LCP. Chen and Xiu [7] improved Burke-Xu's algorithm by simplifying the definition of neighborhood and adding an approximate Newton step to obtain a local quadratic convergence result. Later, Burke and Xu [2] presented two predictor-corrector-type non-interior continuation methods for the monotone LCP and also obtained a local quadratic convergence result. It should be noted that [2, 7] need both the strict complementarity and the uniform nonsingularity assumptions to ensure the local superlinear (quadratic) convergence. To delete the nonsingularity assumption, Tseng [34] studied the local quadratic convergence of general predictor-corrector-type path-following methods for monotone NCPs via the error bound theory. Engelke and Kanzow [13, 14] further investigated the methods given in [34] and proposed two predictor-corrector path-following methods for linear programming. However, the algorithms given in [13, 14, 34] depend strongly on the strict complementarity assumption. Chen et al. [10] discovered the Jacobian consistency property for the Gabrial and Moré smooth function family and first developed a globally and superlinearly convergent smoothing Newton method without strict complementarity. Chen and Chen [6] proposed a non-interior continuation method for $P_0 + R_0$ NCP and monotone NCP with a feasible interior point. By introducing a procedure to update the neighborhoods of the smoothing path, they obtained both global and local superlinear convergence under nonsingularity assumption. Very recently, a class of new smoothing Newton methods were proposed by Qi-Sun-Zhou [29] for NCPs and box constrained variational inequalities. It was shown to possess fast local convergence without the strict complementarity. Very encouraging numerical results of the method were reported in [35]. Due to its simplicity and stronger numerical results, the method has also been used to solve other problems [18, 26, 27, 33]. Among them, Huang et al. [18] proposed a smoothing algorithm for LCP. It was proved to converge to P_0-LCP solution sub-quadratically under a nonsingularity condition and to monotone LCP solution quadratically under strict complementarity. Lastly, Chen and Xiu [5] presented a non-interior one-step continuation method for monotone LCP, which is a modified version of Burke-Xu [2] framework of the non-interior predictor-corrector path-following method. It was proved that Chen-Xiu's algorithm converges globally linearly and locally superlinearly if monotone LCP has strict interior point and nonsingularity assumptions hold for all limit points of its iteration sequence. It should be point out that the nonsingularity assumption used in [5, 6, 18, 29] imply that the solution set is a singleton, but do not imply the strict complementarity holds.

It is worth mentioning that most non-interior continuation methods, e.g., [2,4–7,30], achieve often both global linear and local superlinear (quadratic) convergence. However, only the local superlinear (quadratic) convergence is reachable for most smoothing Newton methods, e.g., [10, 11, 18, 22, 29, 26, 33]. In general, the global linear rate of convergence is not easily obtained without some extra effort [28]. In this paper, we establish a new smoothing Newton method for solving the P_0-LCP (1.1) based on CHKS smoothing function. The proposed algorithm has the following good properties: (i) it needs only to solve a linear system of equations and perform one line search at each iteration. (ii) It is well-defined and we can get a solution of (1.1) from any accumulation point of the iteration sequence generated by the algorithm. In addition, the iteration sequence is bounded if the solution set of (1.1) is nonempty and bounded. (iii) If an accumulation point of the iteration sequence satisfies a nonsingularity assumption, then the whole iteration sequence converges to the accumulation point globally linearly and locally quadratically without the strict complementarity assumption. Note that our algorithm design is based on the Qi-Sun-Zhou algorithmic framework [29] and is motivated by the idea of the algorithm given in [5]. However, different from them, we use a term $\sigma_k \mu_k$ into the perturbed Newton equation. This allows our algorithm to have the convergence result (iii), which is not satisfied for most existing smoothing Newton methods [10, 11, 22, 26, 33]. Hence, compared to some previous literatures, e.g., [5], our smoothing Newton method holds much better convergence results under weaker assumptions. In particular, to the best of our knowledge, this is the first one-step smoothing Newton method for the P_0-LCP to have the convergence property (iii). We implement the proposed algorithm for several standard test problems by a MATLAB code. The preliminary numerical results indicate that the algorithm is promising.

The rest of this paper is organized as follows. We develop a new smoothing Newton method for solving the P_0-LCP (1.1) in the next section. In Section 3, we show its global convergence. In Section 4, it is proved that if an accumulation point of the iteration sequence satisfies a nonsingularity assumption, then the whole iteration sequence converges to the accumulation point globally linearly and locally quadratically in absence of strict complementarity. Some numerical results and conclusions are given in Sections 5 and 6, respectively.

The following notions will be used throughout this paper. All vector are column vectors, the subscript T denotes transpose, R^n (respectively, R) denotes the space of n-dimensional real column vectors (respectively, real numbers), R^n_+ and R^n_{++}

denote the nonnegative and positive orthants of R^n, R_+ (respectively, R_{++}) denotes the nonnegative (respectively, positive) orthant in R. We define $\mathcal{I} := \{1, 2, \cdots, n\}$. For any vector $u \in R^n$, we denote by $diag\{u_i : i \in \mathcal{I}\}$ the diagonal matrix whose ith diagonal element is u_i and $vec\{u_i : i \in \mathcal{I}\}$ the vector u. For simplicity, we use (u, v) for the column vector $(u^T, v^T)^T$. The matrix I represents the identity matrix of arbitrary dimension. The symbol $\|\cdot\|$ stands for the 2-norm. For any $\alpha, \beta \in R_{++}$, $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) means α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$.

2. A New Smoothing Newton Method for P_0-LCP

Our smoothing Newton method is based on Chen-Harker-Kanzow-Smale (CHKS) smoothing function [4, 21, 32] $\phi : R^3 \to R$ defined by

$$\phi(a, b, \mu) = a + b - \sqrt{(a-b)^2 + 4\mu^2}, \qquad \mu > 0. \tag{2.1}$$

Let $z := (x, y, \mu) \in R \times R^{2n}$ and

$$H(z) := H(x, y, \mu) := \begin{pmatrix} Mx + q - y \\ \Phi(x, y, \mu) \\ \mu \end{pmatrix}, \tag{2.2}$$

where

$$\Phi(x, y, \mu) := \begin{pmatrix} \phi(x_1, y_1, \mu) \\ \vdots \\ \phi(x_n, y_n, \mu) \end{pmatrix}.$$

Obviously, $\phi(a, b, 0)$ is just minimum function with the following property

$$\phi(a, b, 0) = 0 \Longleftrightarrow a \geqslant 0, \quad b \geqslant 0, \quad ab = 0. \tag{2.3}$$

Thus, the P_0-LCP (1.1) is equivalent to the following equation

$$H(z) = 0 \tag{2.4}$$

in the sense that their solution sets are coincident.

It is well known that the function $\phi(a, b, \mu)$ is strongly semismooth on R^3, which plays an important role in our local convergence analysis. The following lemma is useful in our later analysis.

LEMMA 2.1. *(a) H is continuously differentiable at any $z = (x, y, \mu) \in R_{++} \times R^{2n}$ with its Jacobian*

$$H'(z) = \begin{pmatrix} M & -I & 0 \\ D_1(z) & D_2(z) & v(z) \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

where

$$\begin{aligned} D_1(z) &:= diag\{1-(x_i-y_i)/\sqrt{(x_i-y_i)^2+4\mu^2} : \quad i \in \mathcal{I}\}, \\ D_2(z) &:= diag\{1+(x_i-y_i)/\sqrt{(x_i-y_i)^2+4\mu^2} : \quad i \in \mathcal{I}\}, \\ v(z) &:= vec\{-4\mu/\sqrt{(x_i-y_i)^2+4\mu^2} : \quad i \in \mathcal{I}\}. \end{aligned}$$

If M is a P_0-matrix, then the matrix $H'(z)$ is nonsingular on $R_{++} \times R^{2n}$.

(b) H is strongly semismooth at any $z \in R^{2n+1}$, and then for any $V \in \partial H(z+h)$, $h \to 0$, it follows that

$$\|H(z+h) - H(z) - Vh\| = O(\|h\|^2).$$

Proof. It is not difficult to see that (b) holds and H is continuously differentiable on $R_{++} \times R^{2n}$. For any $\mu > 0$, a straightforward calculation from (2.2) yields (2.5). Obviously, $0 < (D_1(z))_{ii} < 2$ and $0 < (D_2(z))_{ii} < 2$ for all $i \in \mathcal{I}$. Then we obtain that $D_1(z)$ and $D_2(z)$ are positive diagonal matrices. Since M is a P_0-matrix, by Theorem 3.3 in [4], the matrix $D_1(z) + D_2(z)M$ is nonsingular, which implies that the matrix $H'(z)$ is nonsingular. Therefore, (a) is proved. We complete the proof of this lemma. □

Now we give our one-step smoothing Newton method for the P_0-LCP (1.1)

Algorithm 2.1: (*A new one-step smoothing Newton method*)

Step 0. Choose $\beta, \delta, \sigma \in (0, 1)$ and $\mu_0 \in R_{++}$. Let $\sigma_0 := \min\{\sigma, \mu_0\}$ and $x^0 \in R^n$ be an arbitrary point. Let $y^0 := Mx^0 + q$ and $z^0 := (x^0, y^0, \mu_0)$. Set $k := 0$.

Step 1. If $\|H(z^k)\| = 0$, stop.

Step 2. Compute $\Delta z^k := (\Delta x^k, \Delta y^k, \Delta\mu_k) \in R^{2n+1}$ by

$$H(z^k) + H'(z^k)\Delta z^k = \begin{pmatrix} 0 \\ \sigma_k\mu_k \end{pmatrix}. \tag{2.6}$$

Step 3. Let $\lambda_k := \max\{\delta^m | m = 0, 1, 2, \cdots\}$ such that

$$\|H(z^k + \lambda_k\Delta z^k)\| \leqslant [1 - \beta(1-\sigma_k)\lambda_k]\|H(z^k)\|. \tag{2.7}$$

Step 4. Set $z^{k+1} := z^k + \lambda_k\Delta z^k$ and $k := k + 1$. Go to Step 1.

REMARK. Algorithm 2.1 is motivated by the idea of [5] and is based on the algorithmic framework given in [29]. The main feature of Algorithm 2.1 is that we use the term $\sigma_k = \min\{\sigma, \mu_k\}$ into the perturbed Newton Equation (2.6), which is very different from the algorithms given in [5] and [29]. This allows Algorithm 2.1 to have both global linear and local quadratic convergence without the strict complementarity assumption, which is not satisfied for most existing smoothing Newton methods [10, 11, 22, 26, 29, 33]. In addition, just as the Qi-Sun-Zhou algorithm, Algorithm 2.1 solves only one linear system of Equations (2.6) and performs only one Armijo-type line search (2.7) per iteration.

THEOREM 2.2. *Algorithm 2.1 is well-defined and generates an infinite sequence* $\{z^k = (x^k, y^k, \mu_k)\}$ *with* $0 < \mu_{k+1} < \mu_k \leqslant \mu_0$ *and* $Mx^k + q = y^k$ *for all* $k \geqslant 0$.

Proof. If $\mu_k > 0$, then it follows from Lemma 2.1 (a) that the matrix $H'(z^k)$ is nonsingular. Hence, Step 2 is well-defined at the kth iteration. For any $\lambda \in (0, 1]$, define

$$r(\lambda) := H(z^k + \lambda\Delta z^k) - H(z^k) - \lambda H'(z^k)\Delta z^k. \tag{2.8}$$

It follows from (2.6) that

$$\Delta\mu_k = -(1 - \sigma_k)\mu_k. \tag{2.9}$$

Hence, for any $\lambda \in (0, 1]$, we have

$$\mu_k + \lambda\Delta\mu_k = (1 - \lambda)\mu_k + \lambda\sigma_k\mu_k > 0,$$

which implies from (2.2) and Lemma 2.1 that $H(\cdot)$ is continuously differentiable around z^k. Thus, (2.8) implies that

$$\|r(\lambda)\| = o(\lambda). \tag{2.10}$$

Hence, by (2.6), (2.8) and (2.10), we obtain for any $\lambda \in (0, 1]$ that

$$\begin{aligned}\|H(z^k + \lambda\Delta z^k)\| &\leqslant \|r(\lambda)\| + (1 - \lambda)\|H(z^k)\| + \lambda\sigma_k\mu_k \\ &\leqslant [1 - (1 - \sigma_k)\lambda]\|H(z^k)\| + o(\lambda),\end{aligned}$$

which implies that there exists a constant $\bar{\lambda} \in (0, 1]$ such that

$$\|H(z^k + \lambda\Delta z^k)\| \leqslant [1 - \beta(1 - \sigma_k)\lambda]\|H(z^k)\|$$

holds for any $\lambda \in (0, \bar{\lambda}]$. This shows that Step 3 is well-defined at the kth iteration. Therefore, by (2.9) and Step 3–4, we have $\lambda_k \in (0, 1]$ and

$$\mu_{k+1} = (1 - \lambda_k)\mu_k + \lambda_k\sigma_k\mu_k > 0.$$

On the other hand,

$$\mu_{k+1} \leqslant [1 - (1 - \sigma)\lambda_k]\mu_k < \mu_k.$$

Thus, from $\mu_0 > 0$ and the above statements, we obtain that Algorithm 2.1 is well-defined and generates an infinite sequence $\{z^k = (x^k, y^k, \mu_k)\}$ with $0 < \mu_{k+1} < \mu_k \leqslant \mu_0$ for all $k \geqslant 0$. In addition, by (2.6), we have

$$M\Delta x^k - \Delta y^k = -(Mx^k + q - y^k).$$

Hence, we can complete the proof by the fact $Mx^0 + q = y^0$ and induction on k. □

3. Global Convergence

By Theorem 2.2, Algorithm 2.1 generates an infinite sequence $\{z^k\}$. In this section we show that any accumulation point of the iteration sequence $\{z^k\}$ is a solution of the equation (2.4). Further, if the solution set is nonempty and bounded then the sequence $\{z^k\}$ is bounded and hence has an accumulation point.

ASSUMPTION 3.1. *The solution set of the P_0-LCP (1.1) is nonempty and bounded.*

Note that Assumption 3.1 is the weakest among the existing conditions used to ensure the boundedness of the iteration sequence; see [17].

LEMMA 3.2. *Let $\{z^k = (x^k, y^k, \mu_k)\}$ be the iteration sequence generated by Algorithm 2.1. Then $\{\mu_k\}$ converges to zero.*

Proof. By Theorem 2.2, $\{\mu_k\}$ is monotonically decreasing. Then, there exists $\hat{\mu} \geqslant 0$ such that

$$\lim_{k\to\infty} \mu_k = \hat{\mu}.$$

If $\hat{\mu} = 0$, then the conclusion holds. Suppose that $\hat{\mu} > 0$, then we have $\mu_0 \geqslant \mu_k \geqslant \hat{\mu} > 0$ by Theorem 2.2. In this case, by the definition of CHKS smoothing function and similar to the proof of Proposition 2.1 in [33], we can show that $\|H(z^k)\| \to \infty$ as $\|(x^k, y^k)\| \to \infty$. Hence, the iteration sequence $\{z^k\}$ is bounded and has at least one accumulation point $z^* = (x^*, y^*, \mu_*)$ with $\mu_* = \hat{\mu} > 0$ and $\|H(z^*)\| \geqslant \mu_* > 0$. Subsequently without loss of generality, we may assume that $\{z^k\}$ converges to z^*. Then, it follows from (2.9) and Step 4 that

$$\mu_{k+1} = \mu_k + \lambda_k \Delta z^k < [1 - (1-\sigma)\lambda_k]\mu_k,$$

which implies that $\lim_{k\to\infty} \lambda_k = 0$ by $0 < \sigma < 1$ and $\mu_* > 0$. Set $\tau_k := \lambda_k/\delta$. Then, on one hand, it follows from (2.7) that

$$\|H(z^k + \tau_k \Delta z^k)\| > [1 - \beta(1-\sigma_k)\tau_k]\|H(z^k)\|. \tag{3.1}$$

Since $\|H(z^*)\| > 0$, let $k \to \infty$ in (3.1), we have

$$H(z^*)^T H'(z^*)\Delta z^* \geqslant -\beta(1-\sigma_*)\|H(z^*)\|^2, \tag{3.2}$$

where $\sigma_* = \min\{\sigma, \mu_*\}$. On the other hand, by (2.6), we have

$$H(z^*)^T H'(z^*)\Delta z^* = -\|H(z^*)\|^2 + \sigma_* \mu_*^2 \leqslant -(1-\sigma_*)\|H(z^*)\|^2, \tag{3.3}$$

where the inequality follows from the fact $\|H(z^*)\| \geqslant \mu_*$. It follows from (3.2) and (3.3) that

$$-(1-\sigma_*)\|H(z^*)\|^2 \geqslant -\beta(1-\sigma_*)\|H(z^*)\|^2. \tag{3.4}$$

Since $\|H(z^*)\| > 0$ and $0 < \sigma_* \leqslant \sigma < 1$, (3.4) implies that $\beta \geqslant 1$, which is contracted with the fact $\beta < 1$. Therefore, we have $\mu_* = 0$. That is, μ_k tends to zero as $k \to \infty$. We complete the proof. □

THEOREM 3.3. *Let $\{z^k = (x^k, y^k, \mu_k)\}$ be the iteration sequence generated by Algorithm 2.1. If Assumption 3.1 is satisfied, then $\{z^k\}$ is bounded and hence it has at least one accumulation point $z^* = (x^*, y^*, \mu_*)$ with $H(z^*) = 0$ and (x^*, y^*) is a solution of the P_0-LCP (1.1).*

Proof. Similar to [16], It is not difficult to show that the function $H : R^{2n+1} \to R^{2n+1}$ defined by (2.2) is a weakly univalent function. Because Assumption 3.1 implies that the inverse image $H^{-1}(0)$ is nonempty and bounded. Therefore, by Theorem 2.5 in [31], we obtain that the sequence $\{z^k\}$ is bounded and hence it has at least one limit point, denoted by $z^* = (x^*, y^*, \mu_*)$. Now we need to prove $H(z^*) = 0$. Suppose to the contrary that $H(z^*) \neq 0$. Then we have $\|H(z^*)\| > 0$. By Lemma 3.2, we get $\mu_* = 0$ and hence $\sigma_k \to 0$ as $k \to \infty$. Subsequently, without loss of generality, we assume $\{z^k\}$ converges to z^*. (2.7) implies that $\lim_{k\to\infty} \lambda_k = 0$ by $0 < \beta < 1$ and $\|H(z^*)\| > 0$. Set $\tau_k := \lambda_k/\delta$. Then, on one hand, it follows from (2.7) that

$$\|H(z^k + \tau_k \Delta z^k)\| > [1 - \beta(1 - \sigma_k)\tau_k]\|H(z^k)\|. \tag{3.5}$$

Let $k \to \infty$ in (3.5), by Lemma 2.1 (a), we have

$$H(z^*)^T H'(z^*)\Delta z^* \geqslant -\beta\|H(z^*)\|^2. \tag{3.6}$$

On the other hand, by (2.6), we have

$$H(z^*)^T H'(z^*)\Delta z^* = -\|H(z^*)\|^2. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$-\|H(z^*)\|^2 \geqslant -\beta\|H(z^*)\|^2. \tag{3.8}$$

Since $\|H(z^*)\| > 0$, (3.8) implies that $\beta \geqslant 1$, which is contracted with the fact $\beta < 1$. Therefore, we have $H(z^*) = 0$. Hence z^* is a solution of (2.4), which proves the conclusion. □

4. Global Linear and Local Quadratic Convergence

By Theorem 3.3, we know that Algorithm 2.1 generates a bounded iteration sequence $\{z^k = (x^k, y^k, \mu_k)\}$. Let $z^* = (x^*, y^*, \mu_*)$ be an accumulation point of $\{z^k\}$. Then, it follows from Theorem 3.3 that $\mu_* = 0$ and (x^*, y^*) is a solution of the P_0-LCP (1.1). To establish the rate of convergence for Algorithm 2.1, we assume that (x^*, y^*) satisfies the nonsingularity condition but may not satisfy the strict complementarity. Note that we need the nonsingularity assumption holds for only one accumulation point of the iteration sequence $\{z^k\}$.

THEOREM 4.1. *Suppose that Assumption 3.1 is satisfied and $z^* = (x^*, y^*, \mu_*)$ is an accumulation point of the iteration sequence $\{z^k = (x^k, y^k, \mu_k)\}$ generated by Algorithm 2.1. If all $V \in \partial H(z^*)$ are nonsingular. Then,*

(i) the whole sequence $\{z^k\}$ converges to z^* and $\lambda_k \equiv 1$ for all sufficiently large k;

(ii) $\{\mu_k\}$ and $\{z^k\}$ converge locally quadratically to 0 and z^*, respectively;

(iii) $\{\mu_k\}$ converges globally Q-linearly to zero.

Proof. By Theorem 3.3, $H(z^*) = 0$. Because all $V \in \partial H(z^*)$ are nonsingular, which implies that (1.1) has a unique solution, by Theorem 3.2, the whole sequence $\{z^k\}$ converges to z^*. Moreover, it follows from Proposition 3.1 in [25] that for all z^k sufficiently close to z^*, we have

$$\|H'(z^k)^{-1}\| = O(1). \tag{4.1}$$

From Lemma 2.1 (b), we know that $H(\cdot)$ is strongly semismooth at z^*. Hence, for all z^k sufficiently close to z^*, we have

$$\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| = O(\|z^k - z^*\|^2). \tag{4.2}$$

On the other hand, Lemma 2.1 (b) implies that $H(\cdot)$ is locally Lipschitz continuous near z^*. Therefore, for all z^k sufficiently close to z^*,

$$\|H(z^k)\| = O(\|z^k - z^*\|). \tag{4.3}$$

Thus, (4.3) implies

$$\sigma_k\mu_k = O((\mu_k)^2) = O(\|H(z^k)\|^2) = O(\|z^k - z^*\|^2). \tag{4.4}$$

Then, by (4.1), (4.2) and (4.4), we obtain that

$$\begin{aligned} \|z^k+\Delta z^k-z^*\| &\leqslant \|H'(z^k)^{-1}\|[\|H(z^k)-H(z^*)-H'(z^k)(z^k-z^*)\|+\sigma_k\mu_k] \\ &= O(\|z^k - z^*\|^2). \end{aligned} \tag{4.5}$$

By following the proof of Theorem 3.1 in [24], for all z^k sufficiently close to z^*, we have

$$\|z^k - z^*\| = O(\|H(z^k) - H(z^*)\|). \tag{4.6}$$

Then, because $H(\cdot)$ is strongly semismooth at z^*, for all z^k sufficiently close to z^*, we have

$$\begin{aligned} \|H(z^k + \Delta z^k)\| &= \|H(z^k + \Delta z^k) - H(z^*)\| \\ &= O(\|z^k + \Delta z^k - z^*\|) = O(\|z^k - z^*\|^2) \\ &= O(\|H(z^k) - H(z^*)\|^2) = O(\|H(z^k)\|^2). \end{aligned} \tag{4.7}$$

By Theorem 3.3, $\lim_{k\to\infty} \|H(z^k)\| = 0$. Hence, (4.7) implies that when k sufficiently large, $\lambda_k = 1$ can satisfy (2.7), which proves (i). Therefore, for all z^k sufficiently close to z^*, we have from (i) that

$$z^{k+1} = z^k + \Delta z^k, \quad \mu_{k+1} = \sigma_k\mu_k = O(\mu_k^2),$$

which, together with (4.5), proves (ii).

Next, we prove (iii). By the fact $\sigma_k \leqslant \sigma$ and (i), there exists a positive integer $\bar{k} > 0$ such that

$$\mu_{k+1} = \sigma_k\mu_k \leqslant \sigma\mu_k, \quad \forall k > \bar{k}. \tag{4.8}$$

Since $\hat{\lambda} := \min\{\lambda_k | k \leqslant \hat{k}\} > 0$, we have

$$\mu_{k+1} = [1 - (1 - \sigma_k)\lambda_k]\mu_k \leqslant [1 - (1 - \sigma)\hat{\lambda}/2]\mu_k, \quad \forall k \leqslant \hat{k}. \tag{4.9}$$

Let $C := \max\{\sigma, [1-(1-\sigma)\hat{\lambda}/2]\}$, then we have $C \in (0, 1)$ by the facts $\sigma \in (0, 1)$ and $\hat{\lambda} > 0$. Thus, (4.8) and (4.9) implies

$$\mu_{k+1} \leqslant C\mu_k, \quad \text{for all } k \geqslant 0.$$

Hence, $\{\mu_k\}$ converges globally Q-linearly to zero. □

5. Numerical Results

In this section we present some numerical experiments of Algorithm 2.1 by using a MATLAB code. Throughout the computational experiments, the parameters used in the algorithm were $\beta = 0.25$, $\delta = 0.75$, $\sigma = \mu_0 = 0.0001$. Take $\sigma_k = \mu_k$ for all $k > 0$. The starting point $(x^0, y^0) \in R^{2n}$ has been chosen as follows: let $x^0 \in R^n$ as in the examples and set $y^0 := Mx^0 + q$. In Step 1, we used $\|H(z^k)\| \leqslant 10^{-8}$ as the stopping rule. The numerical results are summarized in Table 1 for different problems. In Table 1, Exam denotes the number of test examples, DIM denotes the number of the variables in the problems, x^0 denotes the starting point, IN denotes the total number of iterations, H0 denotes the value of $\|H(z^0)\|$ and HK denotes the value of $\|H(z^k)\|$ when the algorithm terminates. In the following, we give a brief description of the tested problems.

EXAMPLE 1. (The Murty Problem) This is the fifth example of Kanzow [20] in Section 5 with n variables. The solution is $x^* = (0, \ldots, 0, 1)^T$, $y^* = (1, \ldots, 1, 0)^T$. For this example, Lemke's complementarity pivot algorithm and Cottle and Danzig's principal pivoting method are known to run in exponential time. This example was also tested by Kanzow [21] and Burker and Xu [3]. As in [21, 3], we tested this problem by using $x^0 = (1, \ldots, 1)$ as a starting point. The tested results are listed in Table 1.

EXAMPLE 2. (The Fathi Problem) This is the sixth example of Kanzow [20] in Section 5 with n variables. The solution is $x^* = (1, 0, \ldots, 0)^T$, $y^* = (0, 1, \ldots, 1)^T$. For this example, Lemke's complementarity pivot algorithm and Cottle and Danzig's principal pivoting method are known to run in exponential time. This example was also tested by Kanzow [21] and Burker and Xu [3]. As in [21, 3], we tested this problem by using $x^0 = (1, \ldots, 1)$ as a starting point. The tested results are listed in Table 1.

Table 1. The numerical results of Examples 5.1-5.4

Exam	DIM	x^0	HO	IN	HK
5.1	8	$(1, \dots, 1)$	5.2915	2	6.7370e-016
	16		7.7460	2	9.5390e-016
	32		11.1355	2	1.4260e-015
	64		15.8745	2	2.0350e-015
	128		22.5389	2	2.6618e-015
	256		31.9374	2	3.9968e-015
5.2	8	$(1, \dots, 1)$	5.6569	3	5.0040e-009
	16		8	4	4.6361e-010
	32		11.3137	4	5.0039e-009
	64		16	6	6.4906e-009
	128		22.6274	5	5.0040e-009
	256		32	8	1.2949e-014
5.3	8	$(1, \dots, 1)$	6.6332	2	1.3643e-015
	16		8.7178	2	1.7990e-015
	32		11.8321	2	3.2660e-015
	64		16.3707	2	3.2137e-015
	128		22.8910	2	5.3690e-015
	256		32.1869	2	7.5307e-015
5.4	5	$(0, \dots, 0)$	2	2	1.1952e-014
		$(1, \dots, 1)$	16.3707	2	1.3977e-015
		$(10, \dots, 10)$	153.3754	2	1.4207e-015
		$(100, \dots, 100)$	1.5242e+003	2	1.4207e-015
		$(1000, \dots, 1000)$	1.5233e+004	2	1.4207e-015

EXAMPLE 3. The Ahn Problem with n variables.

$$M = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & -2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & -2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{pmatrix}, \qquad q = -4e.$$

The solution is $x^* = (1.6, 1.3, \dots, 1.3, 1.2, 1, 0.7)^T$, $y^* = (0, \dots, 0)^T$. We tested this problem by using $x^0 = (1, \dots, 1)$ as a starting point. The tested results are listed in Table 1.

EXAMPLE 4. This is the fourth example of Kanzow [20] in Section 5 with 5 variables, which has a nondegenerate solution, namely, $x^* = (1, 0, 1, 1.2, 0.4)^T$,

$y^* = (0, 1, 0, 0, 0)^T$. We used the same starting points as in [20]. The tested results are listed in Table 1.

From Table 1, we can obtain the following observations:

- All problems tested have been solved using only a small number of iterations. Moreover, the numerical results of examples 5.1, 5.2 and 5.4 are significantly better than those appearing in [3, Section 5], [20, Section 7], and [21, Section 5].
- For each problem tested, we tested it by using different starting point or different dimensions of the problem. However, vary of the number of iterations is very small.

Our computational results indicate that the proposed new smoothing Newton-type algorithm works very well for all tested problems in this paper. We expect that the method can be used to solve practical large-scale sparse problems efficiently.

6. Conclusions

Based on the algorithmic framework in [29] and motivated by the idea in [5], we have presented a new one-step smoothing Newton method (Algorithm 2.1) for solving the P_0-matrix linear complementarity problem (1.1). Algorithm 2.1 is simpler than the predictor-corrector-type smoothing algorithms given in [2]. It solves only one linear system of equations and performs only one line search at each iteration. Without the strict complementarity, we have shown that Algorithm 2.1 has both global linear and local quadratic convergence results if the P_0-LCP (1.1) satisfies a nonsingularity condition. In particular, to the best of our knowledge, Algorithm 2.1 is the first one-step smoothing Newton method to have both global linear and local quadratic convergence. Compared to many previous literatures (e.g., [5]), our algorithm has stronger convergence results under weaker assumptions. The preliminary numerical results given in section 5 indicate that the algorithm is promising.

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References

1. Billups, S. C., Dirkse, S. P. and Ferris, M. C. (1997), A comparison of algorithms for large-scale mixed complementarity problems, *Computational Optimization and Applications* 7, pp. 3–25.
2. Burke, J. and Xu, S. (2000), A non-interior predictor-corrector path following algorithm for the monotone linear complementarity problem, *Mathematical Programming* 87, 113–130.
3. Burke, J. and Xu, S. (1998), The global linear convergence of a non-interior path-following algorithm for linear complementarity problems, *Mathematics of Operations Research* 23, 719–734.
4. Chen, B. and Harker, P. T., A non-interior-point continuation method for linear complementarity problems, *SIAM Journal on Matrix Analysis and Applications* 14, 1168-1190.
5. Chen, B. and Xiu, N. (2001), Superlinear noninterior one-step continuation method for monotone LCP in absence of strict complementarity, *Journal of Optimization Theory and Applications*, 108, 317–332.
6. Chen, B. and Chen, X. (1999), A global and local superlinear continuation-smoothing method for $P_0 + R_0$ and monotone NCP, *SIAM Journal on Optimization* 9, 624–645.
7. Chen, B. and Xiu, N. (1999), A global linear and local quadratic non-interior continuation method for nonlinear complementarity problems based on Chen-Mangasarian smoothing functions, *SIAM Journal on Optimization* 9, 605–623.
8. Chen, C. H. and Mangasarian, O. L. (1995), Smoothing methods for convex inequalities and linear complementarity problems, *Mathematical Programming* 71, 51–69.
9. Chen, C. H. and Mangasarian, O. L. (1996), A class of smoothing functions for nonlinear and mixed complementarity problems, *Computational Optimization and Applications* 5, 97–138.
10. Chen, X., Qi, L. and Sun, D. (1998), Global and superlinear convergence of the smoothing Newton method and its application to general box-constrained variational inequalities, *Mathematics of computation* 67, 519–540.
11. Chen, X. and Ye, Y. (1999), On Homotopy-Smoothing Methods for Variational Inequalities, *SIAM Journal on Optimization* 37, 598-616.
12. Cottle, R. W., Pang, J. S. and Stone, R. W. (1992), *The Linear Complementarity Problem*, Academic Press, Boston MA.
13. Engelke, S. and Kanzow, C. (2000), Predictor-corrector smoothing mehtods for the solution of linear programms, preprint 153, Institute of Applied Mathematics, University of Hamburg, Hamburg, March.
14. Engelke, S. and Kanzow, C., Improved smoothing-type mehtods for the solution of linear programms, to appear in *Numerische Mathematik*.
15. Gabrial, S. A. and Moré, J. J. (1996), Smoothing of mixed complementarity problems, in *Complementarity and Variational Problems: State of the Art*, Ferris, M. C. and Pang, J. -S., (eds.), SIAM, Philadelphia, pp. 105–116.
16. Gowda, M. S. and Sznajder, J. J. (1999), Weak univalence and connectedness of inverse images of continuous functions, *Mathematics of Operations Research* 24, 255–261.
17. Huang, Z., Han, J., Xu, D. and Zhang, L. (2001), The non-interior continuation methods for solving the P_0-function nonlinear complementarity problem, *Science in China* 44, 1107–1114.
18. Huang, Z. H., Qi, L. and Sun, D. (2001), Sub-quadratic convergence of a smoothing Newton algorithm for the P_0 – and monotone LCP, Manuscript, August 17.
19. Isac, G. (1993), Complementarity problems, *Lecture Notes in Mathematics* 1528, Springer, Berlin, Heidelberg, New York.
20. Kanzow, C. (1996), Global convergence properties of some iterative methods for linear complementarity problems, *SIAM Journal on Optimization* 6, 326–334.
21. Kanzow, C. (1996), Some noninterior continuation methods for linear complementarity problems, *SIAM Journal on Matrix Analysis and Applications* 17, 851-868.

22. Kanzow, C. and Pieper, H. (1999), Jacobian smoothing methods for nonlinear complementarity problems, *SIAM Journal on Optimization* 9, 342–373.
23. Pardalos, P. M., Ye, Y., Han, C. -G. and Kalinski, J. (1993), Solution of P-matrix linear complementarity problems using a potential reduction algorithm, *SIAM Journal on Matrix Analysis and Applications*, 14, 1048–1060.
24. Qi. L. (1993), Convergence Analysis of Some Algorithms for Solving Nonsmooth Equations, *Mathematics of Operations Research* 18, 227–244.
25. Qi, L. and Sun, J. (1993), A Nonsmooth Version of Newton's Method, *Mathematical Programming* 58, 353–367.
26. Qi, H. (2000), A regularized smoothing Newton method for box constrained variational inequality problems with P_0-functions, *SIAM Journal on Optimization* 10, 315–330.
27. Qi, H. and Liao, L. Z. (1999), A smoothing Newton method for extended vertical linear complementarity problems, *SIAM Journal on Matrix Analysis and Applications* 21, 45–66.
28. Qi, H. and Liao, L. Z. (2000), A smoothing Newton method for general nonlinear complementarity problems, *Computational Optimization and Applications* 17, 231–253.
29. Qi, L., Sun, D. and Zhou, G. (2000), A New Look at Smoothing Newton Methods for Nonlinear Complementarity Problems and Box Constrained Variational Inequality Problems, *Mathematical Programming* 87, 1–35.
30. Qi, L. and Sun, D. (2000), Improving the convergence of non-interior point algorithm for nonlinear complementarity problems, *Mathematics of Computation*, 69, 283-304.
31. Ravindran, G. and Gowda, M. S. (2001), Regularization of P_0-functions in box variational inequality Problems, *SIAM Journal on Optimization* 11, 748–760.
32. Smale, S. (1987), Algorithms for solving equations, in Gleason, A. M. (ed.), *Proceeding of International Congress of Mathematicians*, American Mathematics Society, Providence, RI, pp. 172–195.
33. Sun, D. (1999), A regularization Newton method for solving nonlinear complementarity problems, *Applied Mathematics and Optimization*, 35, 315–339.
34. Tseng, P. (2000), Error bounds and superlinear convergence analysis of some Newton-type methods in optimization, in Di Pillo, G. and Giannessi, F. (eds.), *Nonlinear Optimization and Related Topics*, Kluwer Academic Publishers, Boston, MA, pp. 445-462.
35. Zhou, G., Sun, D. and Qi, L. (1998), Numerical experiences for a class of squared smoothing Newton methods for box constrained variational inequality problems, in Fukushima, M. and Qi, L. (eds.), *Reformulation-Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*, Kluwer Academic Publishers, Boston, MA. pp. 421–441.